Expander Graphs
Exercise Sheet 4

Question 1. Let $A(T_d)$ be the adjacency matrix of the infinite d-regular tree. Show that $\lambda \in \mathrm{spec}(A)$ if and only if $\mathbf{1}_v \notin \mathrm{range}(A - \lambda I)$, where $\mathbf{1}_v$ is the characteristic function of $\{v\}$.

Show further that equation (5.2) has a solution in $\ell_2(V)$ if and only if $|\lambda| > 2\sqrt{d-1}$.

Question 2. Let $\hat{C}_k$ denote the number of sequences $0 = \delta_0, \delta_1, \dots, \delta_{2k} = 0$ such that $\delta_i \in \mathbb{N}_0$, $|\delta_i - \delta_{i-1}| = 1$ for all $i \in [2k-1]$ and exactly j of the δ_i are 0. Show that

$$\hat{C}_k = \binom{2k-j}{k} \frac{j}{2k-j}.$$

Question 3. A graph H is a *lift* of a graph G if there exists a *covering map* $f : V(H) \to V(G)$ such that f maps the neighbourhood $N_H(v)$ of v bijectively to $N_G(f(v))$ for each $v \in V(H)$.

Show that for any eigenfunction $h \in \mathbb{R}^{V(G)}$ of G, $h \circ f$ is an eigenfunction of H with the same eigenvalue. If h is an eigenfunction of H not arising in this manner, show that

$$\sum_{f(x)=v} h(x) = 0 \text{ for all } v \in V(G).$$

Question 4. A lift f is a k-*lift* if each *fibre* $\{x \colon f(x) = v\}$ has size k. Show that H is a k-lift of G if and only if there is a family $\{\pi_{u,v} \colon (u,v) \in \overrightarrow{E}(G)\}$ of permutations in S_n such that $V(H) = V(G) \times [n]$, $\pi_{u,v} = \pi_{v,u}^{-1}$ and

$$E(H) = \{\big((u,i),(v,\pi_e(i))\big) \colon i \in [n], (u,v) \in E(G)\}.$$

If H is a 2-lift of G in the form above, let $\tilde{A}$ be the *signing* of $A(G)$ given by replacing the entries $a_{u,v}$ by -1 if $\pi_{u,v}$ is not the identity. Show that the eigenvalues of H which are not inherited from G as in Question 2 are given by the eigenvalues of $\tilde{H}$.

Question 5. Let $k, d \in \mathbb{N}$ be fixed with $d \geq 3$. Show that whp a random (n,d)-graph has $o(n)$ cycles of length k.